Simulations of Brownian particle motion by the Euler–Maruyama method

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Abstract

A stochastic differential equation (SDE) aims to relate a stochastic process to its composition of random components and base deterministic function. As the relation process is prolonged over time, solutions arise under an initial condition and boundary conditions. Therefore solutions of stochastic differential equations exist and are unique[1]. For this simulation, the Euler–Maruyama (EM) method will be used to approximate and simulate standard Brownian particle motion.

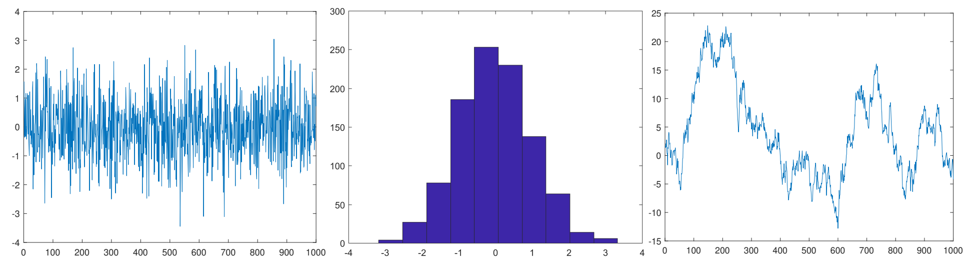
1 Brownian Motion

1.1 Discretized Brownian paths

A scalar *standard Brownian motion* $W(t)$ over $[0, T]$ where $t \in [0, T]$, has conditions

1. $W(0) = 0$
2. For $0 \leq s < t \leq T$, the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$.
3. For $0 \leq s < t < u < v \leq T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

[1] by Existence and Uniqueness Theorem (see app.)



(a) displacement (b) distribution (c) position over time

Figure 1: Normally distributed Brownian motion in 1D with $\sigma = 1$

1.2 Approximation of the Schotastic Integral

Integrating a function h with respect to Brownian motion gives the stochastic integral $\int_0^T h(t)dW(t)$, which can also be written as

$$\sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j)) \tag{1}$$

by the "left" *Riemann sum* (see Appendix). Alternatively, the "midpoint" Riemann sum yields

$$\sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2}\right)(t_{j+1} - t_j). \tag{2}$$

Evaluating the stochastic integrals for each approximation yields exact solutions

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{N-1}\left(W(t_{j+1})^2 - W(t_j)^2 - (W(t_{j+1}) - W(t_j))^2\right) \\ =&\frac{1}{2}\left(W(T)^2 - W(0)^2 - \sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))^2\right) \end{aligned} \tag{3}$$

from (1) and

$$\frac{1}{2}\left(W(T)^2 - W(0)^2\right) + \sum_{j=0}^{N-1}\Delta Z_j(W(t_{j+1}) - W(t_j)), \quad \Delta Z_j = N\left(\frac{\Delta t}{4}\right) \tag{4}$$

from (2). Therefore $\int_0^T h(t)dW(t)$ is defined as the limit of a Riemann sum.

1.3 The Itô integral

As expected, the summation term in (3) will yield value T and variance $O(\delta t)$. Therefore, δt approaches constant T, giving the *Itô integral*

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T. \tag{5}$$

Similarly, (4) gives the *Stratonovich integral*

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2. \tag{6}$$

We will apply the definition of the Itô integral to simulate the solutions of a stochastic differential equation. This is known as the *Euler-Maruyama method.*

2 The Euler-Maruyama method

The solution $X(t)$ to a stochastic differential equation (SDE) satisfies

$$X(t) = X_0 + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s), \quad 0 \leq t \leq T. \tag{7}$$

Notice the right integral uses the Itô integral form. Therefore it is taken with respect to Brownian motion.

2.1 Application of standard Brownian motion

We can then take the SDE form

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad X(0) = X_0, \quad 0 \leq t \leq T, \tag{8}$$

and using the base stochastic process of standard Brownian motion, discretize over the interval $[0, T]$. Thus we can apply the approximations

$$\int_{t_n}^{t_{n+1}} g(s, X(s))dW(s) \approx g(t_n, X_n)\Delta W_n,$$

$$\int_{t_n}^{t_{n+1}} f(s, X(s))ds \approx a(t_n, X_n)\delta t.$$

Then the change dX of the stochastic process over infinitesimal time interval dt can be written

$$dX = \mu(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = X_0, \tag{9}$$

where μ and σ are constants, and $f(X) = \mu X$ and $g(X) = \sigma X$ in (8).

3 Simulations of solutions using the EM method

3.1 Approximate error and distribution

From (9), the exact solution is given by

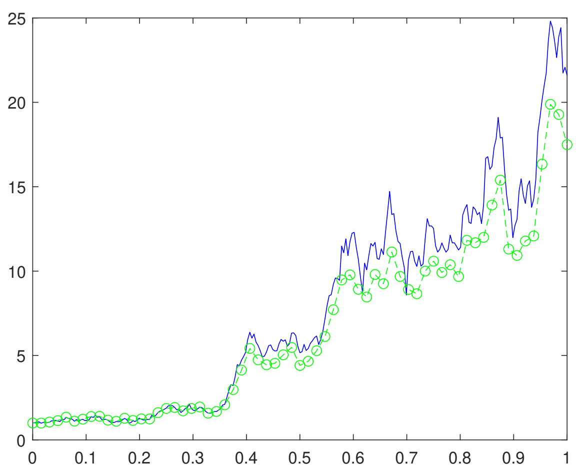
$$X(t) = X_0 e^{\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t}. \tag{10}$$

We can vary the step size to obtain the *approximation error*. To do this we make Z independent runs of the simulation and compute

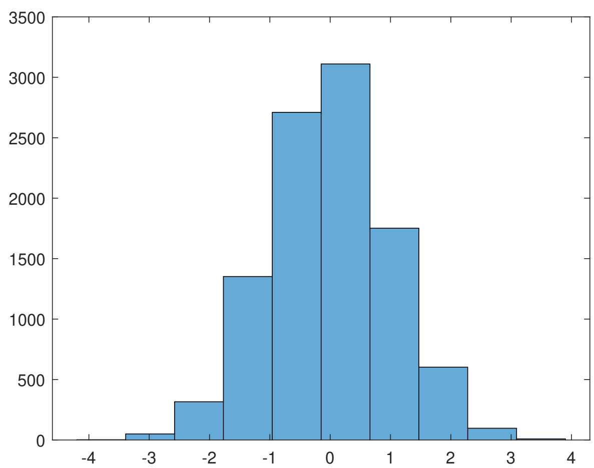
1. $$\frac{1}{Z}\sum_{z=1}^{Z}\left| X_0 e^{\sigma W^z[N] + (\mu - \frac{\sigma^2}{2})T} - X^z[N] \right| \approx \mathbb{E}\left|X_T - X[N]\right| \tag{11}$$

2. $$\left|\mathbb{E}\left(X_0 e^{\sigma W_T + (\mu - \frac{\sigma^2}{2})T}\right) - \frac{1}{Z}\sum_{z=1}^{Z} X^z[N]\right| \approx \left|\mathbb{E}X_T - \mathbb{E}X[N]\right| \tag{12}$$

where N is an independent draw and $N[0..N-1]$ is an array of N independent draws from $N(0,1)$.



(a) Approximation error with 10^4 samples



(b) Distribution over 10 bins

3.2 Mean and variance

The mean of the distribution (b) can be obtained using

$$\bar{x} = N^{-1}\sum x_i.$$

Taking the average difference between the points and $\bar{x}$ gives

$$m_1 = \frac{\sum x_i - \bar{x}}{N}. \tag{13}$$

However, we know that the result will take an approximate value 0 from a condition of normally distributed discretized Brownian paths. Since $\sum \bar{x}$ is represented by N lots of $\bar{x}$,

$$N\bar{x} = N\frac{\sum x_i}{N} = \sum x_i.$$

Thus (13) can be rewritten as

$$\begin{aligned} m_1 &= \frac{\sum x_i - \sum \bar{x}}{N} \\ &= \frac{\sum x_i - \sum x_i}{N} \\ &= 0. \end{aligned}$$

This problem arises since the positive differences above the mean offset the negative differences below it. Thus the mean may be defined as the value in which the differences between the variable values and mean equal zero. Alternatively we may solve for the *average squared difference* instead:

$$m_2 = \frac{\sum (x_i - \bar{x})^2}{N}. \tag{14}$$

This is the *second moment* or variance, and taking $\sqrt{m_2}$ gives the *standard deviation*. We take five intervals from the histogram distribution, compute the mean and variance, and plot as a function of time.

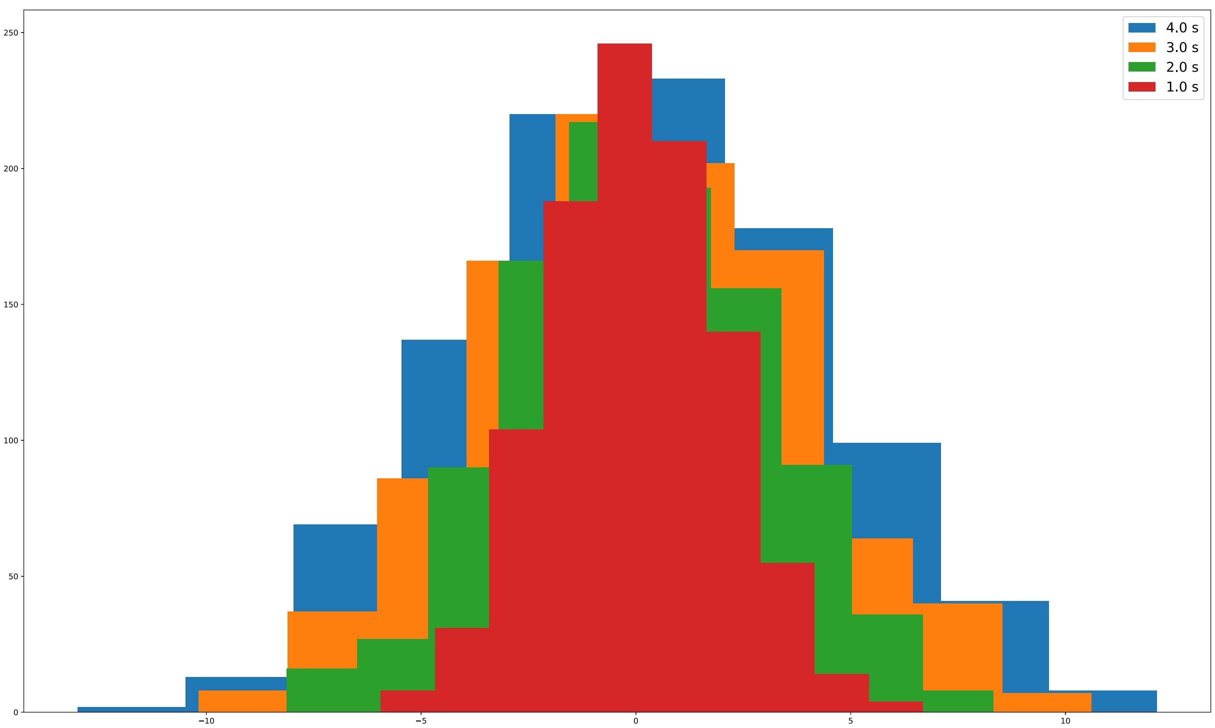


Figure 3: Mean and variance of the distribution

However, the histogram remains uninformative since it is dependent on the choice of bins. We normalize it to obtain the probability distribution,

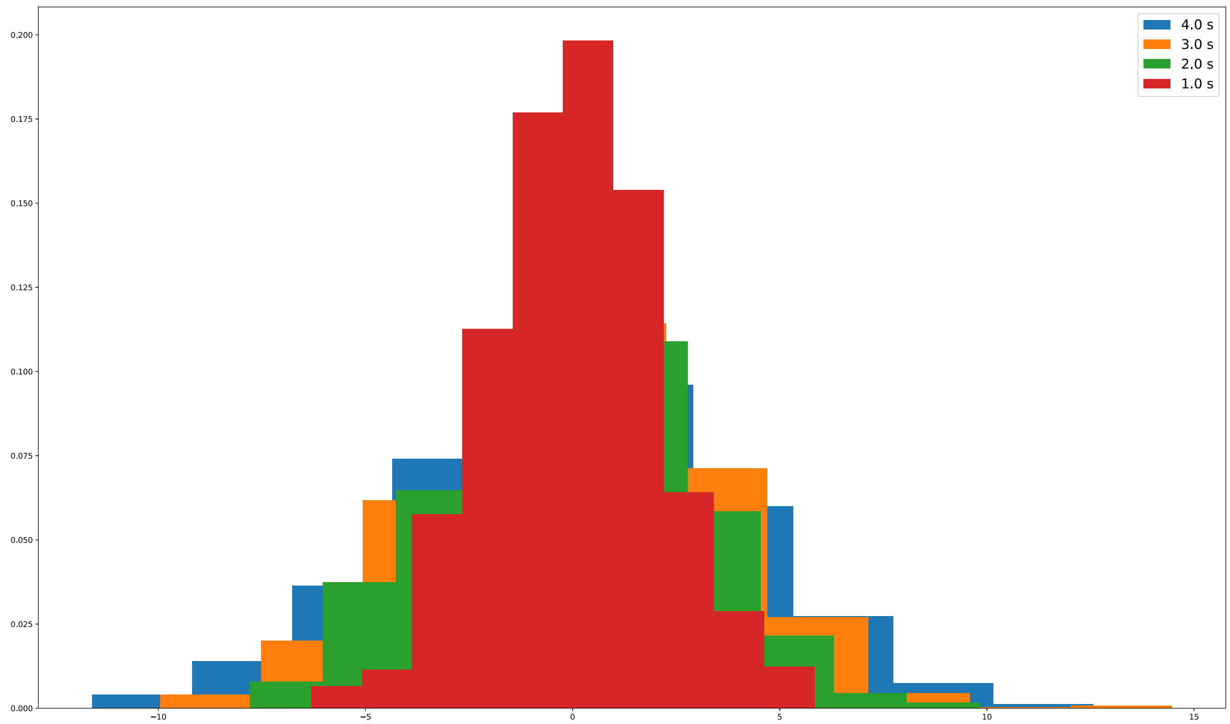


Figure 4: Probability density

4 Comparison to the Diffusion equation

4.1 Solving by superposition

Given the diffusion equation

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho \tag{15}$$

and initial condition

$$\rho(x, 0) = \delta(x - y), \tag{16}$$

we first solve for (15) with

$$\int_{-\infty}^{\infty} |\rho(x,t)|^2 \, dx < \infty, \qquad 0 \leq t. \tag{17}$$

Notice the solution is set to be square integrable to ensure the uniqueness of the solutions. Taking the complex separable solution to (17) gives

$$\rho(x,t) = e^{-k^2 t} e^{ikx} \tag{18}$$

with frequencies k. The solutions can then be combined by Fourier superposition to obtain the general solution

$$\rho(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t} e^{ikx} \hat{\delta}_y(k) dk. \tag{19}$$

Thus the initial condition

$$\rho(x,0) = \delta(x-y) \tag{20}$$

is satisfied, where

$$\hat{\delta}_y(k) = \frac{1}{\sqrt{2\pi}} e^{-iky}. \tag{21}$$

Given the *fundamental solution* equation (see Appendix)

$$L(\rho) = \int_0^l \delta(x-\beta) f(\beta) d\beta = f(x), \tag{22}$$

we obtain the fundamental solution for (17)

$$F(x-y,t) = \frac{1}{2\sqrt{\pi t}} e^{\frac{-(x-y)^2}{4t}}. \tag{23}$$

4.2 Solving numerically

It is also possible to numerically compute the solutions to the diffusion equation. From (15), let

$$\rho(x,0) = \delta(x-0)$$

$$u(x,0) = G(x-0,0,\sigma^2)$$
$$G = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2}\sigma^2}$$

We then iterate over G. Plotting over different time steps gives the diffusion equation model

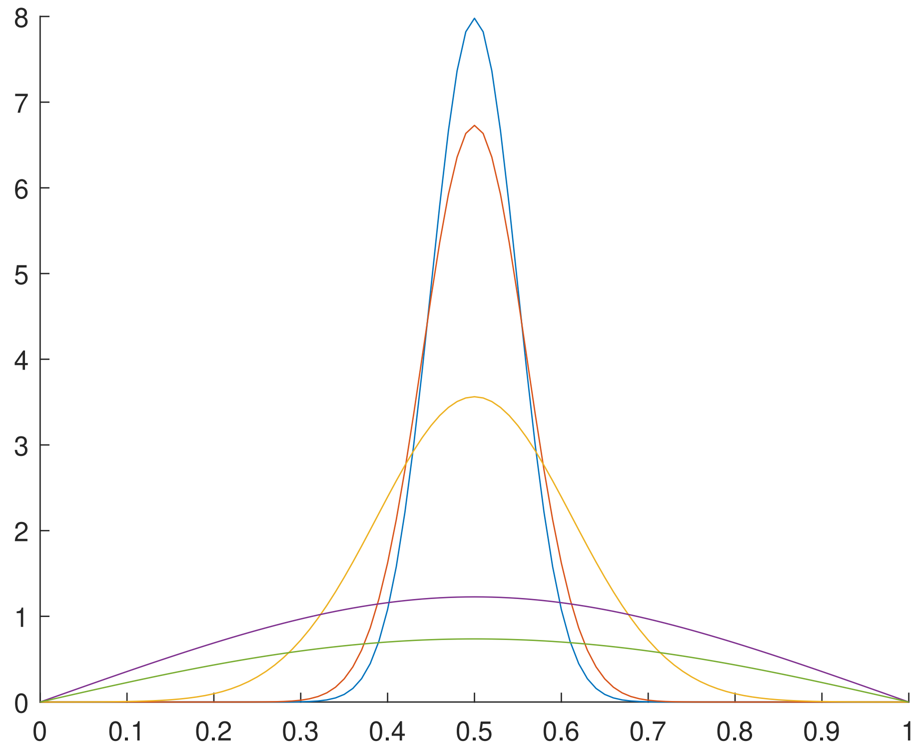


Figure 5: Diffusion over different time steps

Appendices

A Existence-Uniqueness

For the stochastic differential equation

$$dX = G(t, X(t))dt + H(t, X(t))dW(t), \qquad X(t_0) = X_0$$

assume

1. Both $G(t, x)$ and $H(t, x)$ are continuous on $(t, x) \in [t_0, T] \times \mathbb{R}$.
2. The coefficient functions G and H satisfy a Lipschitz condition:
$$|G(t, x) - G(t, y)| + |H(t, x) - H(t, y)| \leq K|x - y|.$$
3. The coefficient functions G and H satisfy a growth condition in the second variable
$$|G(t, x)|^2 + |H(t, x)|^2 \leq K(1 + |x|^2)$$
for all $t \in [t_0, T]$ and $x \in \mathbb{R}$.

Then the stochastic differential equation has a strong solution on $[t_0, T]$ that is continuous with probability 1.

B Riemann sum for stochastic integrals

Given a suitable function h, the integral $\int_0^T h(t)dt$ may be approximated by the Riemann sum

$$\sum_{j=0}^{N-1} h(t_j)(t_{j+1} - t_j), \qquad t_j = j\delta t$$

The integral may be defined by taking the limit $\delta t \to 0$, giving

$$\sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

which is an approximation to a stochastic integral $\int_0^T h(t)dW(t)$.

This section adapted from: "An Algorithmic Introduction to the Numerical Simulation of Stochastic Differntial Equations", by Desmond J. Higham, SIAM Review, Vol. 43

C Fundamental solutions of PDEs

Consider

$$L(u) = \delta(x - \beta), \qquad u(0) = u(l) = 0, \qquad 0 < x < l$$

Let Green's function $u(x, \beta) = G(x, \beta)$ represent the solution. By linearity, we obtain the general solution

$$L(u) = f(x), \qquad u(0) = u(l) = 0, \qquad 0 < x < l$$

as a superposition integral

$$u(x) \equiv \int_0^l G(x, \beta) f(\beta) d\beta.$$

Thus

$$L(u) = \int_0^l \delta(x - \beta) f(\beta) d\beta = f(x).$$

References

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